The Spectrum of an Operator

V. M. Sholapurkar

In what follows, we deal with a bounded linear operator on a complex, separable Hilbert space. Through the set of exercises given below, we attempt to develop a short course in spectral theory.
Note that the space $\mathcal{B}(\mathcal{H})$ of operators on $\mathcal{H}$ is a Banach algebra. An operator T is said to be invertible if there exists an operator S (in $\mathcal{B}(\mathcal{H})$) such that $TS = ST = I$.

1. If T is an operator on a finite dimensional Hilbert space $\mathcal{H}$, then the following statements are equivalent:
   (a) T is invertible
   (b) T is one-one
   (c) T is onto
2. Give an example of an operator which is one-one but not onto.
3. Give an example of an operator which is onto but not one-one.
4. Justify whether true or false: If T is one-one and onto, then T is invertible.
5. Give an example of an operator with infinitely many eigenvalues.
6. Give an example of an operator with no eigenvalue.
7. Prove that an operator T on $\mathcal{H}$ is invertible if and only if the following conditions hold :
   (a) T is bounded from below i.e. there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|, \quad \forall x \in \mathcal{H}$.
   (b) Ran T is dense in $\mathcal{H}$.

8. If both T and T^* are bounded from below, then prove that T^* is invertible.

9. Prove that an operator T is not bounded from below if and only if there exists a sequence of unit vectors x_n such that $\|Tx_n\| \to 0$.

10. Give an example of an operator T for which there is a complex number $\lambda \in \Pi(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded from below}\}$, but λ is not an eigenvalue of T.

11. If T is an isometry, then show that $\Pi(T) \subset \partial\mathbb{D}$, where $\mathbb{D}$ denotes the unit disc.

12. If T is an isometry, show that either $\sigma(T) \subset \partial\mathbb{D}$ or $\sigma(T) = \bar{\mathbb{D}}$.

13. If $\|T\| < 1$, then show that $I - T$ is invertible.

14. Show that the set $\mathcal{J}$ of invertible operators is open in $\mathcal{B}(\mathcal{H})$ and the function $T \to T^{-1}$ is continuous on $\mathcal{J}$.

15. Prove that $\sigma(T)$, the spectrum of T is a closed subset of the disc $\{z \in \mathbb{C} : |z| \leq \|T\|$.

16. Prove that $\sigma(T) \neq \phi$. Further show that $\Pi(T) \neq \phi$.

17. Show that the spectrum of a compact operator K is at most a countable set. Further if $\sigma(K)$ is an infinite set, then K is not invertible.

18. Let $T : l^2 \to l^2$ be given by $T(x_1, x_2, \cdots, x_n, \cdots) = (x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots)$.

    (a) Find eigenvalues of T.

    (b) Show Ran T is dense in l^2.

    (c) Find $\Gamma(T) = \{\lambda \in \mathbb{C} : \text{Ran}(T - \lambda I) \text{ is not dense in } \mathcal{H}\}$.

    (d) Find $\Pi(T)$.

    (e) Find $\sigma(T)$.

19. Repeat exercise 16 for $T : l^2 \to l^2$ given by $T(x_1, x_2, \cdots, x_n, \cdots) = (0, x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots)$.

20. Let (X, Ω, μ) be a $\sigma-$finite measure space and $\mathcal{H} = L^2(X, \Omega, \mu)$. For $\varphi \in L^\infty(X, \Omega, \mu)$, define $M_\varphi : \mathcal{H} \to \mathcal{H}$ by $M_\varphi(f) = \varphi f$. Repeat exercise 16 for M_φ.

21. If $X = [0, 1]$, μ is the Lebesgue measure and φ is continuous, then show that M_φ is not compact.

22. Show that $\Pi_0(T^*) = (\Gamma(T))^*$.

23. If T_n is a sequence of invertible operators and T is non-invertible such that $\|T_n - T\| \to 0$, then show that $0 \in \Pi(T)$.

24. Prove that $\partial\sigma(T) \subset \Pi(T)$.

25. Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be defined by $Te_n = e_{n+1},\ n \in \mathbb{Z}$. Repeat exercise 18.

26. It T is unitary, prove that $\sigma(T) \subset S^1$, the unit circle.

27. If T is normal, prove that $\Pi_0(T) = \Gamma(T)$ and $\sigma(T) = \Pi(T)$.

28. Let $T : l^2 \to l^2$ be given by $Te_n = \dfrac{1}{2^n} e_{n+1}$. Find $\sigma(T)$.

29. Let $k(x, y)$ be a bounded measurable function on $[0, 1] \times [0, 1]$ such that $k(x, y) = 0$ if $x < y$. Define $T : L^2(0, 1) \to L^2(0, 1)$ by $Tf(x) = \int_0^x k(x, y)f(y)dy$. Find $\sigma(T)$.

30. Given a compact subset C of the plane, show that there exists an operator T such that $\sigma(T) = C$.